# VII. *Problems concerning Interpolations. By* Edward Waring, *M. D. F. R. S. and of the Institute of* Bononia, Lucasian *Professor of Mathematics in the University of* Cambridge.

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MR. BRIGGS was the first person, I believe, that invented a method of differences for interpolating logarithms at small intervals from each other: his principles were followed by REGINALD and MOVTON in France. Sir ISAAC NEWTON, from the same principles, discovered a general and elegant solution of the abovementioned problem: perhaps a still more elegant one on some accounts has been since discovered by Messf. NICHOLE and STIRLING. In the following theorems the same problem is resolved and rendered somewhat more general, without having any recourse to finding the successive differences.

## THEOREM I.

Assume an equation $a+bx+cx^2+dx^3 \ldots\ldots x^{n-1}=y$, in which the co-efficients $a$, $b$, $c$, $d$, $e$, &c. are invariable;

let $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. denote $n$ values of the unknown quantity $x$, whofe correfpondent values of $y$ let be reprefented by $s^\alpha$, $s^\beta$, $s^\gamma$, $s^\delta$, $s^\varepsilon$, &c. Then will the equation $a+bx+cx^2+dx^3+ex^4 \ldots x^{n-1}=y=$

$$\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times \&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times \&c.}\times s^\alpha+\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times \&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times \&c.}\times s^\beta$$
$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times \&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times \&c.}\times s^\gamma+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\varepsilon}\times \&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times \&c.}\times s^\delta$$
$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times \&c.}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times \&c.}\times s^\varepsilon+\&c.$$

### DEMONSTRATION.

Write $\alpha$ for $x$ in the equation $y=$
$$\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times \&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times \&c.}\times s^\alpha+\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times \&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times \&c.}\times s^\beta+$$
&c.; and all the terms but the firft in the refulting equation will vanifh, for each of them contains in its numerator a factor $x-\alpha=\alpha-\alpha=0$; and the equation will become $y=\frac{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times \&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times \&c.}\times s^\alpha=s^\alpha$. In the fame manner, by writing $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. fucceffively for $x$ in the given equation it may be proved, that when $x$ is equal to $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. then will $y$ become refpectively $s^\beta$, $s^\gamma$, $s^\delta$, $s^\varepsilon$, which was to be demonftrated.

2. Affume $y=ax^r+bx^{r+s}+cx^{r+2s}+dx^{r+3s}\ldots x^{r+\overline{n-1}s}$; and when $x$ becomes $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. let $y$ become refpectively

ſpectively $s^{\alpha}$, $s^{\beta}$, $s^{\gamma}$, $s^{\delta}$, $s^{\varepsilon}$, &c.; then will $y =$

$$\frac{x^r \times \overline{x^s-\beta^s} \times \overline{x^s-\gamma^s} \times \overline{x^s-\delta^s} \times \overline{x^s-\varepsilon^s} \times \&c.}{\alpha^r \times \overline{\alpha^s-\beta^s} \times \overline{\alpha^s-\gamma^s} \times \overline{\alpha^s-\delta^s} \times \overline{\alpha^s-\varepsilon^s} \times \&c.} \times s^{\alpha}$$

$$+ \frac{x^r \times \overline{x^s-\alpha^s} \times \overline{x^s-\gamma^s} \times \overline{x^s-\delta^s} \times \overline{x^s-\varepsilon^s} \times \&c.}{\beta^r \times \overline{\beta^s-\alpha^s} \times \overline{\beta^s-\gamma^s} \times \overline{\beta^s-\delta^s} \times \overline{\beta^s-\varepsilon^s} \times \&c.} \times s^{\beta}$$

$$+ \frac{x^r \times \overline{x^s-\alpha^s} \times \overline{x^s-\beta^s} \times \overline{x^s-\delta^s} \times \overline{x^s-\varepsilon^s} \times \&c.}{\gamma^r \times \overline{\gamma^s-\alpha^s} \times \overline{\gamma^s-\beta^s} \times \overline{\gamma^s-\delta^s} \times \overline{\gamma^s-\varepsilon^s} \times \&c.} \times s^{\gamma} + \&c.$$

This may be demonſtrated in the ſame manner as the preceding theorem, by writing $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. ſucceſſively for $x$.

### PROBLEM.

Let there be $n$ values $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. of the quantity $x$, to which the $n$ values $s^{\alpha}$, $s^{\beta}$, $s^{\gamma}$, $s^{\delta}$, $s^{\varepsilon}$, &c. of the quantity $y$ correſpond; ſuppoſe theſe quantities to be found by any function X of the quantity $x$; let $\pi$, $\rho$, $\sigma$, $\tau$, &c. be values of the quantities $x$, to which $s^{\pi}$, $s^{\rho}$, $s^{\sigma}$, $s^{\tau}$, &c. values of the quantity $y$ correſpond: for $x$ ſubſtitute its abovementioned values $\pi$, $\rho$, $\sigma$, $\tau$, &c. in the function X, and let the quantities reſulting be $s^{\pi}$, $s^{\rho}$, $s^{\sigma}$, $s^{\tau}$, &c. not equal to the preceding $s^{\pi}$, $s^{\rho}$, $s^{\sigma}$, $s^{\tau}$, &c. reſpectively; to find a quantity which added to the function X ſhall not only give the true values of the quantity $y$ correſponding to the values $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. of the quantity $x$, but alſo

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correſponding to the values $\pi$, $\rho$, $\sigma$, $\tau$, &c. of the above-mentioned quantity $x$.

Aſſume $s^{\pi}-S^{\pi}=T^{\pi}$, $s^{\rho}-S^{\rho}=T^{\rho}$, $s^{\sigma}-S^{\sigma}=T^{\sigma}$, $s^{\tau}-S^{\tau}=T^{\tau}$, &c.; then the errors of the function X will be reſpectively $T^{\pi}$, $T^{\rho}$, $T^{\sigma}$, $T^{\tau}$, &c.; and the correcting quantity ſought may be

$$\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\varepsilon}\times\&c.}\times\frac{\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\tau}\times\&c.}{\overline{\pi-\rho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times\&c.}\times T^{\pi}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\rho-\alpha}\times\overline{\rho-\beta}\times\overline{\rho-\gamma}\times\overline{\rho-\delta}\times\overline{\rho-\varepsilon}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\sigma}\times\overline{x-\tau}\times\&c.}{\overline{\rho-\pi}\times\overline{\rho-\sigma}\times\overline{\rho-\tau}\times\&c.}\times T^{\rho}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times\overline{\sigma-\delta}\times\overline{\sigma-\varepsilon}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\tau}\times\&c.}{\overline{\sigma-\pi}\times\overline{\sigma-\rho}\times\overline{\sigma-\tau}\times\&c.}\times T^{\sigma}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\tau-\alpha}\times\overline{\tau-\beta}\times\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\varepsilon}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\&c.}{\overline{\tau-\pi}\times\overline{\quad-\quad}\times\overline{\tau-\sigma}\times\&c.}\times T^{\tau}$$

$+$ &c.

## *Aliter.*

Let $\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.\times\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\tau}\times\&c.=N$; $\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\varepsilon}\times\&c.\times\overline{\pi-\rho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times\&c.=\Pi$; $\overline{\rho-\alpha}\times\overline{\rho-\beta}\times\overline{\rho-\gamma}\times\overline{\rho-\delta}\times\overline{\rho-\varepsilon}\times\&c.\times\overline{\rho-\pi}\times\overline{\rho-\sigma}\times\overline{\rho-\tau}\times\&c.=P$; $\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times\overline{\sigma-\delta}\times\overline{\sigma-\varepsilon}\times\&c.\times\overline{\sigma-\pi}\times\overline{\sigma-\rho}\times\overline{\sigma-\tau}\times\&c.=\Sigma$; $\overline{\tau-\alpha}\times\overline{\tau-\beta}\times\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\varepsilon}\times\&c.\times\overline{\tau-\pi}\times\overline{\tau-\rho}\times\overline{\tau-\sigma}\times\&c.=T$,&c.; then may the correcting quantity ſought be $N\left(\frac{T^{\pi}}{\Pi\times\overline{x-\pi}}+\frac{T^{\rho}}{P\times\overline{x-\rho}}+\frac{T^{\sigma}}{\Sigma\times\overline{x-\sigma}}+\frac{T^{\tau}}{T\times\overline{x-\tau}}+\&c.\right)$.

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This problem may be demonſtrated in the ſame manner as the preceding theorems, by writing for $x$ in the correcting quantity ſucceſſively its values $\pi$, $\varrho$, $\sigma$, $\tau$, &c.

2. For the correcting quantity ſought may be aſſumed the quantity

$$\frac{\overline{x^s-\alpha^s}\times\overline{x^s-\beta^s}\times\overline{x^s-\gamma^s}\times\overline{x^s-\delta^s}\times \&c.\times x^r\times\overline{x^s-\varrho^s}\times\overline{x^s-\sigma^s}\times\overline{x^s-\tau^s}\times\&c.}{\overline{\pi^s-\alpha^s}\times\overline{\pi^s-\beta^s}\times\overline{\pi^s-\gamma^s}\times\overline{\pi^s-\delta^s}\times \&c.\times \pi^r\times\overline{\pi^s-\varrho^s}\times\overline{\pi^s-\sigma^s}\times\overline{\pi^s-\tau^s}\times\&c.}\times T^{\pi}+\frac{\overline{x^s-\alpha^s}\times\overline{x^s-\beta^s}\times\overline{x^s-\gamma^s}\times\overline{x^s-\delta^s}\times \&c.\times x^r\times\overline{x^s-\pi^s}\times\overline{x^s-\sigma^s}\times\overline{x^s-\tau^s}\times\&c.}{\overline{\varrho^s-\alpha^s}\times\overline{\varrho^s-\beta^s}\times\overline{\varrho^s-\gamma^s}\times\overline{\varrho^s-\delta^s}\times \&c.\times \varrho^r\times\overline{\varrho^s-\pi^s}\times\overline{\varrho^s-\sigma^s}\times\overline{\varrho^s-\tau^s}\times\&c.}\times T^{\varrho}+\&c.$$

3. In general, let z be any quantity which is $=o$, when $x$ becomes either $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c.: let z become ſucceſſively A, B, C, D, &c. when $x$ becomes $\pi$, $\varrho$, $\sigma$, $\tau$, &c. reſpectively. When $x$ either $=\varrho$, $\sigma$, $\tau$, &c. let $\Pi=o$; but if $x=\pi$, let $\Pi=p$: in the ſame manner when $x$ either $=\pi$, $\sigma$, $\tau$, &c. let $P=o$; but when $x=\varrho$ let $P=r$: and ſimilarly, let $\Sigma=o$ when $x$ is either $\pi$, $\varrho$, $\tau$, &c.; but when $x=\sigma$ let $\Sigma=s$: and likewiſe, when $x$ is either $\pi$, $\varrho$, $\sigma$, &c. let $T=o$; but when $x=\tau$ let $T=t$: &c. then for the correcting quantity ſought may be aſſumed $\frac{z}{A}\times\frac{\Pi}{p}\times T^{\pi}+\frac{z}{B}\times\frac{P}{r}\times T^{\varrho}+\frac{z}{C}\times\frac{\Sigma}{s}\times T^{\sigma}+\frac{z}{D}\times\frac{T}{t}\times T^{\tau}+$ &c.

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## THEOREM.

Aſſume $(n)$ quantities $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, &c. then will the ſum of all the $(n)$ quantities of the following kind

$$\frac{\alpha^m}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\text{ \&c.}}+\frac{\beta^m}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\text{ \&c.}}$$

$$+\frac{\gamma^m}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\text{ \&c.}}+\frac{\delta^m}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\text{ \&c.}}$$

$+\dfrac{\varepsilon^m}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\text{ \&c.}}+\text{\&c.}=o$, if $m$ be any whole number leſs than $n-1$; but if $m=n-1$, then will the above mentioned ſum $=1$. In general, the ſum of the $n$ terms

$$\frac{\alpha^m(\beta\gamma\delta\text{ \&c.}+\beta\gamma\varepsilon\text{ \&c.}+\beta\delta\varepsilon+\gamma\delta\varepsilon\text{ \&c.}+\text{\&c.})}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\text{ \&c.}}+$$

$$\frac{\beta^m(\alpha\gamma\delta\text{ \&c.}+\alpha\gamma\varepsilon\text{ \&c.}+\alpha\delta\varepsilon\text{ \&c.}+\gamma\delta\varepsilon\text{ \&c.}+\text{\&c.})}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\text{ \&c.}}+$$

$$\frac{\gamma^m(\alpha\beta\delta\text{ \&c.}+\alpha\beta\varepsilon\text{ \&c.}+\alpha\delta\varepsilon\text{ \&c.}+\beta\delta\varepsilon\text{ \&c.}+\text{\&c.})}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\text{ \&c.}}+$$

$$\frac{\delta^m(\alpha\beta\gamma\text{ \&c.}+\alpha\beta\varepsilon\text{ \&c.}+\alpha\gamma\varepsilon\text{ \&c.}+\text{\&c.})}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\text{ \&c.}}+$$

$$\frac{\varepsilon^m(\alpha\beta\gamma\text{ \&c.}+\alpha\beta\delta\text{ \&c.}+\alpha\gamma\delta\text{ \&c.}+\beta\gamma\delta\text{ \&c.}+\text{\&c.})}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\text{ \&c.}}+\text{\&c.}=o,$$

if $m$ be leſs than $n$, and $m+r$ not equal to $n-1$, where $r$ is equal to the number of letters contained in each of the contents above mentioned $\beta\gamma\delta$, &c. $\beta\gamma\varepsilon$, &c. $\beta\delta\varepsilon$, &c. $\gamma\delta\varepsilon$, &c. &c. &c. reſpectively: but if $m+r=n-1$, then will the above mentioned ſum $=\pm 1$; it will be $+1$ if $r$ be an even number, otherwiſe $-1$.

DEMONSTRATION.

Suppose $a+b\alpha+c\alpha^2+d\alpha^3+e\alpha^4+$ &c. $=\text{S}^\alpha$,
$a+b\beta+c\beta^2+d\beta^3+e\beta^4+$ &c. $=\text{S}^\beta$,
$a+b\gamma+c\gamma^2+d\gamma^3+e\gamma^4+$ &c. $=\text{S}^\gamma$,
$a+b\delta+c\delta^2+d\delta^3+e\delta^4+$ &c. $=\text{S}^\delta$,
$a+b\varepsilon+c\varepsilon^2+d\varepsilon^3+e\varepsilon^4+$ &c. $=\text{S}^\varepsilon$, multiply these equations into A, B, C, D, E, &c. unknown co-efficients to be investigated, and there result

$$\text{A}\times\text{S}^\alpha=\text{A}a+\text{A}b\alpha+\text{A}c\alpha^2+\text{A}d\alpha^3+\text{A}e\alpha^4+\&c.$$
$$\text{B}\times\text{S}^\beta=\text{B}a+\text{B}b\beta+\text{B}c\beta^2+\text{B}d\beta^3+\text{B}e\beta^4+\&c.$$
$$\text{C}\times\text{S}^\gamma=\text{C}a+\text{C}b\gamma+\text{C}c\gamma^2+\text{C}d\gamma^3+\text{C}d\gamma^4+\&c.$$
$$\text{D}\times\text{S}^\delta=\text{D}a+\text{D}b\delta+\text{D}c\delta^2+\text{D}d\delta^3+\text{D}d\delta^4+\&c.$$
$$\text{E}\times\text{S}^\varepsilon=\text{E}a+\text{E}b\varepsilon+\text{E}c\varepsilon^2+\text{E}d\varepsilon^3+\text{E}e\varepsilon^4+\&c.\ \&c.\ \&c.$$

Now suppose $\text{AS}^\alpha+\text{BS}^\beta+\text{CS}^\gamma+\text{DS}^\delta+\text{ES}^\varepsilon+$ &c. $=a+bx+cx^2+dx^3+ex^4+$ &c. and the correspondent parts respectively equal to each other; that is, $a\,(\text{A}+\text{B}+\text{C}+\text{D}+\text{E}+\&c.)=a$; $b\,(\text{A}\alpha+\text{B}\beta+\text{C}\gamma+\text{D}\delta+\text{E}\varepsilon+\&c.)=bx$; $\text{A}\alpha^2+\text{B}\beta^2+\text{C}\gamma^2+\text{D}\delta^2+\text{E}\varepsilon^2+\&c.=x^2$; $\text{A}\alpha^3+\text{B}\beta^3+\text{C}\gamma^3+\text{D}\delta^3+\text{E}\varepsilon^3+\&c.=x^3$; $\text{A}\alpha^4+\text{B}\beta^4+\text{C}\gamma^4+\text{D}\delta^4+\text{E}\varepsilon^4+\&c.=x^4$, &c.: But it follows from Theorem I. that (if $\text{AS}^\alpha+\text{BS}^\beta+\text{CS}^\gamma+\text{DS}^\delta+\text{ES}^\varepsilon+\&c.=a+bx+cx^2+dx^3+ex^4+\&c.$) $\text{A}=\dfrac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}$,

$$B=\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.},\quad C=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.},$$

$$D=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\varepsilon}\times\&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\&c.},\quad E=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\&c.}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\&c.},$$

&c.: ſubſtitute theſe values for A, B, C, D, E, &c. reſpectively in the preceding equations ($A+B+C+D+E+\&c.=1$, $A\alpha+B\beta+C\gamma+D\delta+E\varepsilon+\&c.=x$, $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+E\varepsilon^2+\&c.=x^2$, $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\varepsilon^3+\&c.=x^3$, &c.) and there reſult the equations (1) $\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}$

$$+\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}+\&c.=1;$$

$$(2)\ \alpha\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}+\beta\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}$$

$$+\gamma\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}+\&c.=x;$$

$$(3)\ \alpha^2\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}+\beta^2\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}$$

$+\gamma^2\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}=x^2$; and in general,

$$\alpha^m\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}+\beta^m\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}$$

$$+\gamma^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}+\delta^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\varepsilon}\times\&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\&c.}$$

$+\&c.=x^m$, whatever may be the values of the quantities $x$; $\alpha, \beta, \gamma, \delta, \varepsilon$, &c.: reduce all theſe fractions into terms, proceeding according to the dimenſions of the quantity $x$, and it is evident, that the ſum of all the fractions multiplied

tiplied into any dimenſion of $x$ not equal to $m$ will be $= 0$; but the ſum of all the fractions multiplied into $x^m$ will be $= 1$: from this propoſition the theorem is eaſily deduced.

I have invented and demonſtrated from different principles to the preceding the firſt part of this theorem, a particular caſe of which was publiſhed by me many years ago.

From this theorem may eaſily be deduced ſeveral others of a ſimilar nature.